

GIBBS-TYPE MEASURES FOR THE NON-PERIODIC TWO-DIMENSIONAL EULER EQUATION

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ABSTRACT. We define Gaussian invariant measures for the two-dimensional Euler equation in the non-periodic setting and show the existence of its solution with initial conditions on the support of the measures. Uniqueness and continuity of the velocity flow are proved.

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1. INTRODUCTION

Euler equation describes the time evolution of an incompressible non-viscous fluid with constant density. This fundamental equation has been and still is intensively studied. Among the numerous references on the Euler equation, we cite the books [5, 8, 9]. It is known that solutions do not blow up starting from smooth data with finite kinetic energy (T. Kato (1967), C. Bardos (1967) among others). In two dimensions and when the initial vorticity is bounded, existence, uniqueness and global regularity of solutions was shown (V.I. Yudovich, 1963); these results were extended, in the framework of weak solutions, to the case where the initial vorticity belongs to L^p , with $p > 1$ and even for $p = 1$, when the vorticity is some finite measure.

A more geometric approach, identifying the solutions of the Euler equation with velocities of geodesics in a space of diffeomorphisms of the underlying state space, was initiated by V. Arnold (1966). It allowed to show existence of local solutions in some Sobolev spaces (Ebin and Marsden, 1970).

Much less is known about irregular solutions of the Euler equation. This paper is devoted to a class of such solutions.

In statistical approaches to hydrodynamics, discussed in the physics literature on turbulence, one considers the evolution of probability densities instead of pointwise solutions. A major subject of interest is the search for invariant measures. In particular such measures are important because they can be used to prove the existence and study the properties of Euler flows defined almost-everywhere with respect to them.

In this paper we extend the work [1] in two dimensions to the non-periodic setting. We prove the existence of invariant probability measures for the Euler flow and show the existence of these flows, for all times and in some weak sense, living in the support of the invariant measures. Those are spaces of very low regularity, namely Sobolev spaces of negative order.

2. 2D EULER EQUATIONS

Consider the incompressible non-viscous Euler equations on \mathbb{R}^2

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0 \quad (1)$$

where $u : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denotes the time dependent velocity field and $p : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the pressure. The first equation is Newton's second law (the acceleration is proportional to the pressure) and the second equation is the incompressibility condition.

Let $\nabla^\perp \varphi = (-\partial_2 \varphi, \partial_1 \varphi)$ where ∂_1, ∂_2 denote respectively the partial derivative with respect to the first and second variable.

Theorem 2.1. *The time dependent vector field u is a smooth solution of (1) if and only if there exists a smooth (real) function φ (stream function) such that $u = \nabla^\perp \varphi$ and φ is a solution of the equation*

$$\frac{\partial \Delta \varphi}{\partial t} = (\nabla^\perp \varphi \cdot \nabla) \Delta \varphi. \quad (2)$$

Proof. We refer to [2], for example. □

The two problems, (1) and (2), are equivalent; below we consider (2).

2.1. Periodic case. We recall here the most relevant results from [1] about the periodic case. On the space $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2 \simeq [0, L]^2$, consider equation (2) with periodic boundary condition

$$\varphi(0, y, t) = \varphi(L, y, t) \text{ and } \varphi(x, 0, t) = \varphi(x, L, t), \quad \forall (x, y) \in \mathbb{T}^2.$$

In [1] is considered the case $L = 2\pi$, but the analysis for general $L > 0$ is identical if we simply re-scale.

The energy and the enstrophy, namely $E = \frac{1}{2} \int_{\mathbb{T}^2} |u|^2 dx$ and $S = \frac{1}{2} \int_{\mathbb{T}^2} |\text{curl} u|^2 dx$, are conserved by the Euler velocity. In terms of the stream function φ we have

$$E = -\frac{1}{2} \int_{\mathbb{T}^2} \varphi \Delta \varphi dx$$

and

$$S = \frac{1}{2} \int_{\mathbb{T}^2} |\Delta \varphi|^2 dx.$$

We denote by $\{e_k^L\}_{k \in \mathbb{Z}^2}$ the following orthonormal basis of $L^2(\mathbb{T}^2)$,

$$e_k^L = \frac{1}{L} e^{i \frac{2\pi}{L} k \cdot x}, \quad \forall k \in \mathbb{Z}^2.$$

For all $\varphi \in L^2(\mathbb{T}^2)$ we have

$$\varphi(x, t) = \sum_{k > 0} \varphi_k^L(t) e_k^L(x),$$

and we can identify the Sobolev spaces $H^\beta(\mathbb{T}^2)$ defined by

$$H^\beta(\mathbb{T}^2) := \{u : \mathbb{T}^2 \rightarrow \mathbb{R} : (I - \Delta)^{\beta/2} u \in L^2(\mathbb{T}^2)\}$$

with

$$H^\beta := \left\{ u = \sum_{k > 0} u_k^L e_k^L : \sum_{k > 0} \left(\frac{2\pi k}{L} \right)^{2\beta} |u_k^L|^2 \right\} \quad (3)$$

where we say that $k = (k_1, k_2) \in \mathbb{Z}^2$ is positive if $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$ and where $k^2 = k_1^2 + k_2^2$.

For all β , H^β is a Hilbert space with inner product given by

$$\langle u, v \rangle_\beta := \sum_{k>0} \left(\frac{2\pi k}{L} \right)^{2\beta} u_k^L \bar{v}_k^L.$$

By means of the basis expansion, on $[0, L]^2$, the equations reduce to an infinite dimensional system of first order ODEs

$$\frac{d}{dt} \varphi_k^L(t) = B_k^L(\varphi), \quad \forall k \in \mathbb{Z}^2 \quad (4)$$

where

$$B_L(\varphi) := \sum_{k>0} B_k^L(\varphi) e_k^L(x) \quad (5)$$

and

$$B_k^L(\varphi) = \frac{1}{L} \left(\frac{2\pi}{L} \right)^2 \sum_{\substack{h>0 \\ h \neq k}} \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2} \right] \varphi_h^L \varphi_{k-h}^L, \quad (6)$$

where $h^\perp = (-h_2, h_1)$. We write $B_k^L(\varphi) = \sum_h \alpha_{h,k}^L \varphi_h^L \varphi_{k-h}^L$, with

$$\alpha_{h,k}^L = \frac{1}{L} \left(\frac{2\pi}{L} \right)^2 \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2} \right]. \quad (7)$$

2.2. Notations. Let us consider some relevant function spaces that will be used below. We define the local Sobolev space $H_{loc}^\beta(\mathbb{R}^2)$ by

$$H_{loc}^\beta(\mathbb{R}^2) := \{u : \forall K \text{ compact}, (I - \Delta)^{\beta/2} u \in L^2(K)\}.$$

For negative or non-integer values of β , the operator $(I - \Delta)^\beta$ is considered as a pseudo-differential operator. We may assume that the compact sets K are of the type $K = [0, L] \times [0, L]$ for $L \in \mathbb{N}^*$. It is not possible to define a norm on $H_{loc}^\beta(\mathbb{R}^2)$, but we can equip this space with the topology induced by the distance d_β defined by

$$d_\beta(u, v) := \sum_{L \in \mathbb{N}^*} 2^{-L} \frac{\|(I - \Delta)^{\beta/2}(u - v)\|_{L^2([0, L]^2)}}{1 + \|(I - \Delta)^{\beta/2}(u - v)\|_{L^2([0, L]^2)}}. \quad (8)$$

In particular the metric space $(H_{loc}^\beta(\mathbb{R}^2); d_\beta)$ is complete. For further results concerning local Sobolev spaces we refer to [4]. Analogously we define $W_{loc}^{\beta, \infty}(\mathbb{R}^2)$ by

$$W_{loc}^{\beta, \infty}(\mathbb{R}^2) := \{u : \forall K \text{ compact}, (I - \Delta)^{\beta/2} u \in L^\infty(K)\}.$$

The space $W_{loc}^{\beta, \infty}(\mathbb{R}^2)$ is complete if endowed with the distance d_∞ defined by

$$d_\infty(u, v) := \sum_{L \in \mathbb{N}^*} 2^{-L} \frac{\|(I - \Delta)^{\beta/2}(u - v)\|_{L^\infty([0, L]^2)}}{1 + \|(I - \Delta)^{\beta/2}(u - v)\|_{L^\infty([0, L]^2)}}.$$

Clearly we have

$$W_{loc}^{\beta, \infty}(\mathbb{R}^2) \subseteq H_{loc}^\beta(\mathbb{R}^2).$$

For each $L \in \mathbb{N}^*$, the norm $\|(I - \Delta)^{\beta/2} u\|_{L^p([0, L]^2)}$ is equivalent to the norm $\|D^\beta u\|_{L^p([0, L]^2)}$ for every $\beta \in \mathbb{R}$ and $1 \leq p \leq +\infty$, thus it is possible to define other distances \tilde{d}_β and \tilde{d}_∞ such that $(H_{loc}^\beta(\mathbb{R}^2), \tilde{d}_\beta)$ and $(W_{loc}^{\beta, \infty}(\mathbb{R}^2), \tilde{d}_\infty)$ are still complete. Indeed we have

$$\begin{aligned} d_\beta(u, v) &= \sum_{L \in \mathbb{N}^*} 2^{-L} \frac{\|(I - \Delta)^{\beta/2}(u - v)\|_{L^2([0, L]^2)}}{1 + \|(I - \Delta)^{\beta/2}(u - v)\|_{L^2([0, L]^2)}} \\ &\leq \sum_{L \in \mathbb{N}^*} 2^{-L} C(L) \frac{\|D^\beta(u - v)\|_{L^2([0, L]^2)}}{1 + \|D^\beta(u - v)\|_{L^2([0, L]^2)}} =: \tilde{d}_\beta(u, v) \end{aligned}$$

and

$$d_\infty(u, v) \leq \sum_{L \in \mathbb{N}^*} 2^{-L} C_\infty(L) \frac{\|D^\beta(u - v)\|_{L^\infty([0, L]^2)}}{1 + \|D^\beta(u - v)\|_{L^\infty([0, L]^2)}} =: \tilde{d}_\infty(u, v),$$

where $C(L)$ and $C_\infty(L)$ are constants at most proportional to L^η for some $\eta \in \mathbb{R}$.

We say that a function u belongs to the weighted Sobolev space $W^{\beta, \infty}(\mathbb{R}^2, 1 + |x|)$ for some fixed $\beta \in \mathbb{R}$ if

$$\|(1 + |x|)^{-1} D^\beta u\|_{L^\infty(\mathbb{R}^2)} < +\infty.$$

Whenever β is negative or is not an integer the operator D^β is understood as a pseudo-differential operator. The following inclusion holds:

$$W^{\beta, \infty}(\mathbb{R}^2, 1 + |x|) \subseteq W_{loc}^{\beta, \infty}(\mathbb{R}^2). \quad (9)$$

Below, we use $X \lesssim Y$ to denote the estimate $X \leq CY$ for some constant C . Unless stated otherwise C is an unessential constant, in particular independent from the period L . Also we always denote by $C(L)$ a constant with growth at most on powers of L , that is $C(L) \sim L^\eta$ with $\eta \in \mathbb{R}$.

3. THE INVARIANT MEASURES

In the periodic setting and for each parameter $\gamma \in \mathbb{R}^+$, invariant probability measures $\mu_{L, \gamma}$ were constructed (see [1]). In this section we define measures μ_γ as the weak limits of $\mu_{L, \gamma}$ when L tends to infinity. The support of μ_γ is $H_{loc}^\beta(\mathbb{R}^2)$.

3.1. Approximations of μ_γ . On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider the stochastic process defined by

$$\Phi_{L, R}(\omega, x) := \sum_{\substack{k > 0 \\ k_1 < L^2 r_1 \\ k_2 < L^2 r_2}} a_k^L(\omega) e_k^L(x)$$

where

$$a_k^L(\omega) := \delta_k(\omega) \sqrt{\frac{2}{\gamma}} \left(\frac{L}{2\pi k} \right)^2,$$

$$\delta_k(\omega) := W_{k^2+1}(\omega) - W_{k^2}(\omega)$$

and where $\{W_{k^2}\}_{k \in \mathbb{Z}^2}$ is a complex-valued Brownian motion. For all fixed k , a_k^L is a complex-valued Gaussian random variable with mean zero and variance $\frac{2}{\gamma} \left(\frac{L}{2\pi k} \right)^4$. Therefore $\Phi_{L, R}$ is a Gaussian vector with law and covariance matrix given respectively by,

$$(\det M(L))^{-1/2} e^{-\langle a, M(L)^{-1} a \rangle} \prod_{\substack{k > 0 \\ k_1 < L^2 r_1 \\ k_2 < L^2 r_2}} \gamma \frac{da_k^L(\omega)}{2\pi}$$

and

$$M(L)_{k, j} = \mathbb{E} \mathbb{P}(a_k^L \bar{a}_j^L) = \delta_j^k \frac{2}{\gamma} \left(\frac{L}{2\pi k} \right)^4.$$

Here δ_j^k denotes the Kronecker symbol and we have,

$$\langle a, M(L)^{-1} a \rangle = \sum_{\substack{k > 0 \\ k_1 < L^2 r_1 \\ k_2 < L^2 r_2}} \left(\frac{2}{\gamma} \left(\frac{L}{2\pi k} \right)^4 \right)^{-1} |a_k^L(\omega)|^2.$$

Remark that, if

$$u(x) = \sum_{\substack{k > 0 \\ k_1 < L^2 r_1 \\ k_2 < L^2 r_2}} u_k^L e_k^L(x),$$

then

$$\sum_{\substack{k>0 \\ k_1 < L^2 r_1 \\ k_2 < L^2 r_2}} \left(\frac{2}{\gamma} \left(\frac{L}{2\pi k} \right)^4 \right)^{-1} |u_k^L|^2 = \frac{\gamma}{2} \int_{\mathbb{T}^2} |\Delta u|^2 dx;$$

that is

$$\langle u, M(L)^{-1} u \rangle = S(u),$$

where by $S(u)$ we denote the enstrophy. Hence the measure $d\mu_{L,\gamma}$, formally defined by

$$d\mu_{L,\gamma}(u) := e^{-\frac{\gamma}{2} \int_{\mathbb{T}^2} |\Delta u|^2 dx} \mathcal{D}u, \quad \mathcal{D}u = \prod_{k>0} \gamma \left(\frac{2\pi k}{L} \right)^4 \frac{du_k^L}{2\pi} \quad (10)$$

is the law of Φ_L on some Banach space, where

$$\Phi_L(\omega, x) := \sum_{k>0} a_k^L(\omega) e_k^L(x). \quad (11)$$

The measure $\mu_{L,\gamma}$ coincides with the Gibbs-type measure, relative to the enstrophy, defined in [1]. It was shown in [1] that $(H^\beta, H^2, \mu_{L,\gamma})$ is a complex abstract Wiener space for $\beta < 1$; that is H^2 is a densely embedded Hilbert subspace of the Banach space H^β and $\mu_{L,\gamma}$ is a Gaussian measure since

$$\int e^{i\gamma l(u)} d\mu_{L,\gamma}(u) = e^{-\frac{1}{2}\gamma \|l\|_2^2}, \quad \forall l \in (H^\beta)' \subset H^2.$$

The space H^β denotes the support of $\mu_{L,\gamma}$ and H^2 the associated Cameron-Martin space.

We define Φ as the limit in $L^2(\Omega; H_{loc}^\beta(\mathbb{R}^2))$ of the sequence of random variables $\{\Phi_L\}_{L \in \mathbb{N}^*}$ given in equation (11) and we define the measure μ_γ on functions of \mathbb{R}^2 as the image measure under the random variable Φ . In the next proposition we follow the ideas of [7] where the Klein-Gordon equation on the real line is considered.

Proposition 3.1. *The sequence $\{\Phi_L\}_{L \in \mathbb{N}^*}$ is a Cauchy sequence in $L^2(\Omega; H_{loc}^\beta(\mathbb{R}^2))$ for $\beta < 1$.*

Proof. First observe that

$$W^{\beta,\infty}(\mathbb{R}^2) \subseteq W^{\beta,\infty}(\mathbb{R}^2, 1 + |x|) \subseteq W_{loc}^{\beta,\infty}(\mathbb{R}^2) \subseteq H_{loc}^\beta(\mathbb{R}^2)$$

and that we can write for $0 < L < S$

$$\Phi_L - \Phi_S = \Phi_L - \Phi_{L,R} + \Phi_{L,R} - \Phi_{S,R} + \Phi_{S,R} - \Phi_S.$$

We will show that $\mathbb{E}_{\mathbb{P}} \|D^\beta(\Phi_L - \Phi_{L,R})\|_{L^\infty(\mathbb{R}^2)}^2$ converges to zero when R tends to infinity uniformly in L and that $\mathbb{E}_{\mathbb{P}} \|(1 + |x|)^{-1} D^\beta(\Phi_{L,R} - \Phi_{S,R})\|_{L^\infty(\mathbb{R}^2)}^2$ tends to zero when L tends to

infinity uniformly in R . We have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}} \|D^{\beta}(\Phi_L - \Phi_{L,R})\|_{L^{\infty}(\mathbb{R}^2)}^2 &= \mathbb{E}_{\mathbb{P}} \left[\sup_{x \in \mathbb{R}^2} \left| \sum_{\substack{k_1 \geq L^2 r_1 \\ k_2 \geq L^2 r_2}} \left(\frac{2\pi k}{L} \right)^{\beta-2} \delta_k(\omega) \sqrt{\frac{2}{\gamma}} e_k^L(x) \right| \right]^2 \\
&\leq \mathbb{E}_{\mathbb{P}} \left[\frac{1}{L} \sum_{\substack{k_1 \geq L^2 r_1 \\ k_2 \geq L^2 r_2}} \left(\frac{2\pi k}{L} \right)^{\beta-2} |\delta_k(\omega)| \sqrt{\frac{2}{\gamma}} \right]^2 \\
&= \frac{1}{L^2} \frac{2}{\gamma} \sum_{\substack{k_1 \geq L^2 r_1 \\ k_2 \geq L^2 r_2}} \sum_{\substack{h_1 \geq L^2 r_1 \\ h_2 \geq L^2 r_2}} \left(\frac{2\pi k}{L} \right)^{\beta-2} \left(\frac{2\pi h}{L} \right)^{\beta-2} \mathbb{E}_{\mathbb{P}} [\delta_k(\omega) \bar{\delta}_h(\omega)] \\
&\leq \frac{2}{\gamma} \sum_{\substack{k_1 \geq L^2 r_1 \\ k_2 \geq L^2 r_2}} \left(\frac{2\pi k}{L} \right)^{2\beta-4} \\
&\lesssim \int_{[R, +\infty)^2} \frac{dy}{y^{4-2\beta}} \leq \varepsilon
\end{aligned}$$

for R sufficiently big and uniformly in L , since $\beta < 1$.

Now suppose that $L = 2^n$ and $S = 2^m$ with $n \leq m$; we have

$$D^{\beta}(\Phi_{2^n, R} - \Phi_{2^m, R}) = \sqrt{\frac{2}{\gamma}} \left[\sum_{\substack{k > 0 \\ k_1 < 2^{2n} r_1 \\ k_2 < 2^{2n} r_2}} \left(\frac{2\pi k}{2^n} \right)^{\beta-2} \delta_k(\omega) 2^{-n} e^{i \frac{2\pi}{2^n} k \cdot x} \right. \quad (12)$$

$$\left. - \sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} \left(\frac{2\pi l}{2^m} \right)^{\beta-2} \delta_l(\omega) 2^{-m} e^{i \frac{2\pi}{2^m} l \cdot x} \right]. \quad (13)$$

Moreover we have

$$\delta_l(\omega) 2^{-m} \simeq W_{\frac{l^2+1}{2^{2m}}}(\omega) - W_{\frac{l^2}{2^{2m}}}(\omega) =: \varepsilon_{2^{-2m}, l^2}(\omega),$$

where here \simeq denotes the symbol of identification in law, and that $\varepsilon_{2^{-2m}, l^2}(\omega)$ can be written as

$$\varepsilon_{2^{-2m}, l^2}(\omega) = \sum_{j=0}^{2^{2(n-m)}-1} \varepsilon_{2^{-2n}, 2^{2(n-m)} l^2 + j}(\omega). \quad (14)$$

Indeed

$$\begin{aligned}
\sum_{j=0}^{2^{2(n-m)}-1} \varepsilon_{2^{-2n}, 2^{2(n-m)} l^2 + j}(\omega) &= \sum_{j=0}^{2^{2(n-m)}-1} W_{\frac{2^{2(n-m)} l^2 + j + 1}{2^{2n}}}(\omega) - W_{\frac{2^{2(n-m)} l^2 + j}{2^{2n}}}(\omega) \\
&= W_{\frac{l^2+1}{2^{2m}}}(\omega) - W_{\frac{l^2}{2^{2m}}}(\omega) \\
&= \varepsilon_{2^{-2m}, l^2}(\omega).
\end{aligned}$$

Therefore

$$\begin{aligned} D^\beta(\Phi_{2^n, R} - \Phi_{2^m, R}) &\simeq \\ &\simeq \sqrt{\frac{2}{\gamma}} \sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} \sum_{j=0}^{2^{2(n-m)}-1} \varepsilon_{2^{-2n}, 2^{2(n-m)}l^2+j}(\omega) \left[\frac{e^{i2\pi \frac{(2^{2(n-m)}l+j) \cdot x}{2^n}}}{\left(\frac{2^{2(n-m)}l+j}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi \frac{l}{2^m} \cdot x}}{\left(\frac{l}{2^m}\right)^{2-\beta}} \right]. \end{aligned}$$

where we write $2^{(n-m)}l + j := (2^{(n-m)}l_1 + j; 2^{(n-m)}l_2 + j)$ for any $l = (l_1, l_2) \in \mathbb{Z}^2$ and $j \in \{0, \dots, 2^{(n-m)}-1\}$. To get the last equality (in law) we used: in (12) the change of variable $k = 2^{2(n-m)}l + j$ and the fact that both the random variables $\varepsilon_{2^{-2n}, (2^{2(n-m)}l+j)^2}$ and $\varepsilon_{2^{-2n}, 2^{2(n-m)}l^2+j}$ are distributed as a Gaussian with mean zero and variance 2^{-2n} ; in (13) the replacement of (14).

Take the $L^2(\Omega)$ norm of $D^\beta(\Phi_{2^n, R} - \Phi_{2^m, R})$:

$$\mathbb{E}_{\mathbb{P}} |D^\beta(\Phi_{2^n, R} - \Phi_{2^m, R})|^2 \lesssim \sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} \sum_{j=0}^{2^{2(n-m)}-1} 2^{-2n} \left[\frac{e^{i2\pi \frac{(2^{2(n-m)}l+j) \cdot x}{2^n}}}{\left(\frac{2^{2(n-m)}l+j}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi \frac{l}{2^m} \cdot x}}{\left(\frac{l}{2^m}\right)^{2-\beta}} \right]^2$$

and use that the directional derivatives of the function $y \in \mathbb{R}^2 \mapsto \frac{e^{i2\pi y \cdot x}}{y^{2-\beta}}$ are bounded by $C(\beta) \frac{(1+2\pi|x|)}{y^{2-\beta}}$ in order to obtain

$$\begin{aligned} &\sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} \sum_{j=0}^{2^{2(n-m)}-1} 2^{-2n} \left[\frac{e^{i2\pi \frac{(2^{2(n-m)}l+j) \cdot x}{2^n}}}{\left(\frac{2^{2(n-m)}l+j}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi \frac{l}{2^m} \cdot x}}{\left(\frac{l}{2^m}\right)^{2-\beta}} \right]^2 \\ &\lesssim \sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} \sum_{j=0}^{2^{2(n-m)}-1} 2^{-2n} \frac{(1+2\pi|x|)^2}{\left(\frac{l}{2^m}\right)^{4-2\beta}} \left(\frac{j}{2^n}\right)^2. \end{aligned}$$

Use the inequality

$$\sum_{j=0}^{2^{2(n-m)}-1} \left(\frac{j}{2^n}\right)^2 \leq \sum_{j=0}^{2^{2(n-m)}} (2^{-2m})^2 = 2^{2n-6m}$$

to get

$$\sum_{\substack{l > 0 \\ l_1 < 2^{2m} r_1 \\ l_2 < 2^{2m} r_2}} 2^{-6m} \frac{(1+2\pi|x|)^2}{\left(\frac{l}{2^m}\right)^{4-2\beta}} \lesssim \varepsilon(1+|x|)^2 \int_{[a, +\infty)^2} \frac{dy}{y^{4-2\beta}} \lesssim \varepsilon(1+|x|)^2$$

for m sufficiently big and uniformly in R since $\beta < 1$ and $a \in (0, 1)$. Back to L and S we have $\mathbb{E}_{\mathbb{P}} \|(1+|x|)^{-1} D^\beta(\Phi_{L, R} - \Phi_{S, R})\|_{L^\infty(\mathbb{R}^2)}^2 \leq \varepsilon$ for L sufficiently big and uniformly in R . \square

In the following we denote by μ_γ the law of Φ where Φ is the limit of $\{\Phi_L\}_{L \in \mathbb{N}^*}$ in $L^2(\Omega; H_{loc}^\beta(\mathbb{R}^2))$. This L^2 -convergence implies that $\mu_{L, \gamma}$ converges weakly to μ_γ in $H_{loc}^\beta(\mathbb{R}^2)$ when L tends to infinity.

3.2. Support of μ_γ . Here we study the support of the measure μ_γ . Since μ_γ is the law of Φ , its support is defined as the space in which $\Phi(\omega, \cdot)$ takes values \mathbb{P} -almost surely.

Proposition 3.2. *Let $\beta < 1$, we have*

$$\text{supp}(\mu_\gamma) = H_{loc}^\beta(\mathbb{R}^2).$$

Proof. We have

$$\mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi, 0) \leq \mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi, \Phi_{L,R}) + \mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi_{L,R}, 0),$$

where d_{β} denotes the metric for $H_{loc}^{\beta}(\mathbb{R}^2)$ defined in (8). On one hand and by Proposition 3.1, $\mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi, \Phi_{L,R})$ tends to zero when L and R tend to infinity. On the other $\mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi_{L,R}, 0) \leq C < +\infty$ since we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} d_{\beta}(\Phi_{L,R}, 0) &\leq \sum_L 2^{-L} C(L) \mathbb{E}_{\mathbb{P}} \|D^{\beta} \Phi_{L,R}\|_{L^2([0,L]^2)} \\ &\lesssim \sum_L 2^{-L} C(L) < +\infty. \end{aligned}$$

We used the fact that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \|D^{\beta} \Phi_{L,R}\|_{L^2([0,L]^2)}^2 &= \sum_{\substack{k>0 \\ k_1 < L^2 R_1 \\ k_2 < L^2 R_2}} \left(\frac{2\pi k}{L} \right)^{2\beta} \mathbb{E}_{\mathbb{P}} |a_k^L(\omega)|^2 \\ &\lesssim \sum_{\substack{k>0 \\ k_1 < L^2 R_1 \\ k_2 < L^2 R_2}} \left(\frac{k}{L} \right)^{2\beta-4} \\ &\lesssim \int_{[a,+\infty)^2} \frac{dy}{y^{4-2\beta}} \leq C < +\infty \end{aligned}$$

for $a > 0$ small enough; and that $C(L)$ depends on the period as previously explained in Subsection 2.2. \square

Formally the measure μ_{γ} is given by

$$d\mu_{\gamma}(\varphi) = \frac{1}{Z} e^{-\frac{\gamma}{2} \int_{\mathbb{R}^2} |\Delta \varphi|^2 dx} \mathcal{D}\varphi \quad (15)$$

where Z is a suitable renormalizing constant. For all fixed $L \in \mathbb{N}^*$, the measure μ_{γ} on functions restricted to the compact phase space $[0, L]^2$ is in fact the measure $\mu_{L,\gamma}$. As in [1] for $(H^{\beta}, H^2, \mu_{L,\gamma})$ we can show that $(H_{loc}^{\beta}(\mathbb{R}^2), H_{loc}^2(\mathbb{R}^2), \mu_{\gamma})$ is a complex abstract Wiener space for $\beta < 1$.

4. THE VELOCITY FLOW ON \mathbb{R}^2

The aim of this section is to prove global existence and uniqueness of the Euler flow on the set under which μ_{γ} is invariant.

4.1. Approximations of the vector field B . We start by recalling some properties of the vector field B_L in the periodic setting, given by equations (5)-(6) and previously derived in [1].

Proposition 4.1. *The vector field B_L is divergence-free with respect to the measure $\mu_{L,\gamma}$, that is $\delta_{\mu_{L,\gamma}} B_L = 0$.*

Proof. We refer to [1] and only remark that the conservation of the enstrophy is essential to prove the statement. \square

We recall the proof of the $L_{\mu_{L,\gamma}}^2$ -regularity of B_L , as we are interested in the dependence on the period L of such estimates. For further details see [1] or [6].

Proposition 4.2. *The vector field $B_L \in L_{\mu_{L,\gamma}}^2(H^{\beta}; H^{\beta})$ for all $\beta < -1$.*

Proof.

$$\mathbb{E}_{\mu_{L,\gamma}} \|B_L(\varphi)\|_{H^{\beta}}^2 = \mathbb{E}_{\mu_{L,\gamma}} \sum_{k>0} \left(\frac{2\pi k}{L} \right)^{2\beta} |B_k^L(\varphi)|^2$$

From $B_k^L(\varphi) = \sum_h \alpha_{h,k}^L \varphi_h^L \varphi_{k-h}^L$ we have that

$$\begin{aligned} \mathbb{E}_{\mu_{L,\gamma}} |B_k^L(\varphi)|^2 &= \sum_{h,h'} \alpha_{h,k}^L \alpha_{h',k}^L \mathbb{E}_{\mu_{L,\gamma}} (\varphi_h^L \varphi_{k-h}^L \varphi_{h'}^L \varphi_{k-h'}^L) \\ &= 2 \sum_h |\alpha_{h,k}^L|^2 \mathbb{E}_{\mu_{L,\gamma}} (|\varphi_h^L|^2) \mathbb{E}_{\mu_{L,\gamma}} (|\varphi_{k-h}^L|^2) \\ &\lesssim \sum_h |\alpha_{h,k}^L|^2 \frac{L^8}{h^4 (k-h)^4} \\ &\leq L^2 \sum_h \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2} \right]^2 \frac{1}{h^4 (k-h)^4} \leq L^2 C. \end{aligned} \quad (16)$$

Then

$$\mathbb{E}_{\mu_{L,\gamma}} \|B_L(\varphi)\|_{H^\beta}^2 \lesssim \frac{1}{L^{2\beta-2}} \sum_{k>0} \frac{1}{k^{-2\beta}}$$

converges for $\beta < -1$. □

Remark 4.1. For the vector field on $[0, L]^2$ the expression $B_L(\varphi) = \sum_k B_k^L(\varphi) e_k^L(x)$ where B_k^L is defined in (6) is valid. Note however that the Euler vector field does not depend on L ; it is the same on every finite phase space approximation and thus B_L trivially converges to B , the Euler vector field on \mathbb{R}^2 , when L goes to infinity.

Next we show that $B : H_{loc}^\beta(\mathbb{R}^2) \rightarrow H_{loc}^\beta(\mathbb{R}^2)$ is regular with respect to $L_{\mu_\gamma}^2$.

Corollary 4.1. For $\beta < -1$, $B \in L_{\mu_\gamma}^2(H_{loc}^\beta(\mathbb{R}^2); H_{loc}^\beta(\mathbb{R}^2))$.

Proof. We show that $E_{\mu_\gamma} d_\beta(B(\varphi), 0) < +\infty$ where d_β denotes the metric for $H_{loc}^\beta(\mathbb{R}^2)$ defined in (8). We have

$$\begin{aligned} E_{\mu_\gamma} d_\beta(B(\varphi), 0) &= \sum_{L \in \mathbb{N}^*} 2^{-L} C(L) E_{\mu_\gamma} \frac{\|B(\varphi)\|_{H^\beta}}{1 + \|B(\varphi)\|_{H^\beta}} \\ &\leq \sum_{L \in \mathbb{N}^*} 2^{-L} C(L) E_{\mu_{L,\gamma}} \|B_L(\varphi)\|_{H^\beta} \end{aligned}$$

where $E_{\mu_{L,\gamma}} \|B_L(\varphi)\|_{H^\beta} \leq CL^{2-2\beta}$ since $\beta < -1$ and because of (16). Hence

$$E_{\mu_\gamma} d_\beta(B(\varphi), 0) \lesssim \sum_{L \in \mathbb{N}^*} 2^{-L} C(L) L^{2-2\beta} < +\infty. \quad \square$$

4.2. Existence. For any fixed $L \in \mathbb{N}^*$ and $R \in \mathbb{N}^2$ we consider a phase space projection on $[0, L]^2$ and a finite dimensional approximation of equation (2). Therefore we study the following system of ODEs for all $k \in \mathbb{Z}^2$ with $k > 0$, $k_1 < L^2 R_1$ and $k_2 < L^2 R_2$:

$$\begin{aligned} \frac{d}{dt} U_k^{L,R}(t, \varphi^{L,R}) &= B_k^{L,R}(U^{L,R}(t, \varphi^{L,R})) \\ U_k^{L,R}(0, \varphi^{L,R}) &= \varphi_k^{L,R} \end{aligned}$$

for

$$\varphi^{L,R}(t, x) = \sum_{\substack{k>0 \\ k_1 < L^2 R_1 \\ k_2 < L^2 R_2}} \varphi_k^{L,R}(t) e_k^L(x) \in \mathbb{C}^d,$$

where $d = d(R) := \#\{k \in \mathbb{Z}^2 : k > 0 \text{ and } k_i < L^2 R_i \text{ for } i = 1, 2\}$ and where

$$B_k^{L,R}(\varphi^{L,R}) = \frac{1}{L} \left(\frac{2\pi}{L} \right)^2 \sum_{\substack{h > 0 \\ h \neq k \\ h_1 < L^2 R_1 \\ h_2 < L^2 R_2}} \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2} \right] \varphi_h^{L,R} \varphi_{k-h}^{L,R}.$$

From the regularity of the finite dimensional quadratic vector field $B^{L,R}$ we know that there exists an associated global flow on \mathbb{C}^d , that is for all positive $k \in \mathbb{Z}^2$ with $k_1 < L^2 R_1$ and $k_2 < L^2 R_2$ we have

$$U_k^{L,R}(t, \varphi^{L,R}) = \varphi_k^{L,R} + \int_0^t B_k^{L,R}(U_k^{L,R}(s, \varphi^{L,R})) ds, \quad \forall t \in \mathbb{R}.$$

For $\varphi^L \in H^\beta$ we write

$$\varphi^L = \Pi_R \varphi^L + \Pi_R^\perp \varphi^L = \varphi^{L,R} + \Pi_R^\perp \varphi^L$$

where Π_R is the orthogonal projection on the subspace spanned by $\{e_k : k > 0 \text{ and } k_i < L^2 R_i \text{ for } i = 1, 2\}$. Also we define

$$U_k^{L,R}(t, \varphi^L) := U_k^{L,R}(t, \varphi^{L,R}) + \Pi_R^\perp \varphi^L,$$

then $U^{L,R}(t, \varphi^L)$ is in fact a $B^{L,R}$ -flow on $H^\beta([0, L]^2)$. Finally for $\varphi \in H_{loc}^\beta(\mathbb{R}^2)$ we write

$$\varphi = \varphi|_{[0, L]^2} + \varphi|_{[0, L]^2^c} = \varphi^L + \varphi|_{[0, L]^2^c}$$

and we define

$$U_k^{L,R}(t, \varphi) := U_k^{L,R}(t, \varphi^L) + \varphi|_{[0, L]^2^c};$$

it follows that $U^{L,R}(t, \varphi)$ is in fact a $B^{L,R}$ -flow on $H_{loc}^\beta(\mathbb{R}^2)$. Furthermore we have

$$U^{L,R}(t, \varphi) = \sum_{\substack{k > 0 \\ k_1 < L^2 R_1 \\ k_2 < L^2 R_2}} U_k^{L,R}(t, \varphi) e_k^L$$

with $U_k^{L,R}(\cdot, \varphi) \in C(\mathbb{R}; \mathbb{C})$.

Next we prove the existence, in the sense of the following theorem, of a flow for (2) taking values in $H_{loc}^\beta(\mathbb{R}^2)$ for $\beta < -1$.

Theorem 4.1. *Let $\beta < -1$. There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a globally defined flow $\tilde{U}(\cdot, \tilde{\omega}) \in C(\mathbb{R}; H_{loc}^\beta(\mathbb{R}^2))$ for $\tilde{\omega} \in \tilde{\Omega}$, such that*

(i)

$$\tilde{U}(t, \tilde{\omega}) = \tilde{U}(0, \tilde{\omega}) + \int_0^t B(\tilde{U}(s, \tilde{\omega})) ds, \quad \tilde{P} - a.e. \tilde{\omega}, \quad \forall t \in \mathbb{R}$$

(ii) and

$$\int f(\tilde{U}(t, \tilde{\omega})) d\tilde{P}(\tilde{\omega}) = \int f(\varphi) d\mu_\gamma(\varphi), \quad \forall f \in C_b$$

Proof. Consider $U_k^{L,R}$ as a stochastic process with law on $C(\mathbb{R}; \mathbb{C})$ defined by

$$\nu_k^{L,R}(\Gamma) := \mu_\gamma\{\varphi \in H_{loc}^\beta(\mathbb{R}^2) : U_k^{L,R}(\cdot, \varphi) \in \Gamma\}, \quad \Gamma \subset C(\mathbb{R}; \mathbb{C}).$$

The measure $\mu_\gamma^{L,R}$ is invariant under $U^{L,R}$ as a consequence of Proposition 4.1; this, together with the fact that $\mu_\gamma^{L,R}$ is weakly convergent, implies that the law of $U_k^{L,R}$ weakly converges to some ν_k in $C(\mathbb{R}; \mathbb{C})$. Therefore by Skorohod's theorem there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and two stochastic processes $\tilde{U}^{L,R}, \tilde{U}$ with laws respectively $\mu_\gamma^{L,R}, \mu_\gamma$, such that $\tilde{U}^{L,R}(\cdot, \tilde{w})$ converges to $\tilde{U}(\cdot, \tilde{w})$ \tilde{P} -a.e. \tilde{w} when L, R tend to infinity. Statement (ii) of the theorem follows. Moreover for all $L \in \mathbb{N}^*$ we have

$$\int \sum_k \left(\frac{2\pi k}{L} \right)^{2\beta} |\tilde{U}_k^L(t, \tilde{w})|^2 d\tilde{P}(\tilde{w}) = \int \|\varphi^L\|_{H^\beta}^2 d\mu_{L,\gamma}(\varphi^L) \leq C < +\infty$$

for $\beta < -1$; this implies that $\tilde{U}(t, \tilde{\omega})$ takes values in $H_{loc}^\beta(\mathbb{R}^2)$ for all $t \in \mathbb{R}$.

Finally, in order to prove (i) we have to check that

$$\mathbb{E}_{\tilde{P}} d_\beta \left(\int_0^t [B_k^{L,R}(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}(s, \tilde{\omega}))] ds; 0 \right)$$

tends to 0 when L and R tend to infinity. We have

$$\begin{aligned} \mathbb{E}_{\tilde{P}} d_\beta \left(\int_0^t [B_k^{L,R}(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}(s, \tilde{\omega}))] ds; 0 \right) &\leq \\ \mathbb{E}_{\tilde{P}} d_\beta \left(\int_0^t [B_k^{L,R}(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}^{L,R}(s, \tilde{\omega}))] ds; 0 \right) &+ \\ + \mathbb{E}_{\tilde{P}} d_\beta \left(\int_0^t [B_k(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}(s, \tilde{\omega}))] ds; 0 \right). \end{aligned}$$

The first term is bounded by

$$\sum_{L \in \mathbb{N}^*} 2^{-L} C(L) \sum_k \left(\frac{2\pi k}{L} \right)^{2\beta} \int_0^t \mathbb{E}_{\tilde{P}} |B_k^{L,R}(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}^{L,R}(s, \tilde{\omega}))|^2 ds.$$

It converges to 0 when L and R tend to infinity by the invariance of the measure and the L^2 convergence of $B_k^{L,R}$ towards B_k . Analogously the second term is bounded by

$$\sum_{L \in \mathbb{N}^*} 2^{-L} C(L) \sum_k \left(\frac{2\pi k}{L} \right)^{2\beta} \int_0^t \mathbb{E}_{\tilde{P}} |B_k(\tilde{U}^{L,R}(s, \tilde{\omega})) - B_k(\tilde{U}(s, \tilde{\omega}))|^2 ds.$$

This term also converges to 0 when L and R go to infinity by the equi-integrability of the functions $B_k(\tilde{U}^{L,R}(s, \tilde{\omega}))$ and the convergence of the flows $\tilde{U}^{L,R}(s, \tilde{\omega})$ towards $\tilde{U}(s, \tilde{\omega})$ (similar to the arguments used in [1]). □

4.3. Uniqueness. Every time that we consider the vorticity equation projected on the torus a uniqueness argument, similar to the one presented in [3], applies. Uniqueness of the velocity flow follows from uniqueness of its law seen as the solution of the corresponding continuity equation; as in the classical DiPerna Lions approach for vector fields with low regularity, see [?]. We use the machinery from [3], namely Theorem 4.7, to say that the law of \tilde{U}^L is a Dirac measure on the trajectories, the proof of this relies on the fact that the solution of the continuity equation is unique. However we are in a very particular case: B_L is autonomous, quadratic and divergence-free. The latter hypothesis permit to show uniqueness in a simpler way than the one presented in [3], in particular we do not need any additional assumption on the gradient of B_L .

Let k_t^L be the Radon-Nikodym density of $d(\tilde{U}^L(t, \cdot) * \tilde{P})$ with respect to $d\mu_{L,\gamma}$ at time $t \in \mathbb{R}$. We have that k_t^L is a bounded weak solution of

$$\begin{aligned} \frac{d}{dt} k_t^L(\varphi) &= - \langle B_L(\varphi), \nabla k_t^L(\varphi) \rangle_\beta, \quad \text{in } \mathbb{R}^+ \times H^\beta; \\ k_0^L(\varphi) &= 1; \end{aligned} \tag{17}$$

that is

$$\int_0^\infty \int_{H^\beta} k_t^L(\varphi) (-\partial_t f + \langle B_L(\varphi), \nabla f \rangle_\beta) d\mu_{L,\gamma}(\varphi) dt = \int_{H^\beta} f(0, \varphi) d\mu_{L,\gamma}(\varphi), \quad \forall f \in \mathcal{D}_t, \tag{18}$$

where \mathcal{D}_t denotes the space of differentiable functions on $\mathbb{R}^+ \times H^\beta$ depending on a finite number of coordinates. Clearly $k_t^L \equiv 1$ is a solution of (17), below we show that it is unique. Remark that for each Galerkin approximation of B_L , B_L^n with $n \in \mathbb{N}$, uniqueness holds since B_L^n is quadratic. Thus $k_t^{L,n} \equiv 1$ is the unique solution of the truncated continuity equation. Now, let

\tilde{k}_t^L be another solution of (17), that is \tilde{k}_t^L verifies (18). We have

$$\begin{aligned} & \int_0^\infty \int_{H^\beta} \tilde{k}_t^L(\varphi) (-\partial_t f + \langle B_L(\varphi), \nabla f \rangle_\beta) d\mu_{L,\gamma}(\varphi) dt - \int_{H^\beta} f(0, \varphi) d\mu_{L,\gamma}(\varphi) \\ &= \int_0^\infty \int_{H^\beta} \tilde{k}_t^L(\varphi^n) (-\partial_t f + \langle B_L(\varphi^n), \nabla f \rangle_\beta) d\mu_{L,\gamma}^n(\varphi^n) dt \int_{H^\beta} d\mu_{L,\gamma}^{n,\perp}(\varphi^{n,\perp}) \\ & \quad - \int_{H^\beta} f(0, \varphi^n) d\mu_{L,\gamma}^n(\varphi^n) \int_{H^\beta} d\mu_{L,\gamma}^{n,\perp}(\varphi^{n,\perp}) \\ &= \int_0^\infty \int_{H^\beta} (-\partial_t f + \langle B_L(\varphi), \nabla f \rangle_\beta) d\mu_{L,\gamma}(\varphi) dt - \int_{H^\beta} f(0, \varphi) d\mu_{L,\gamma}(\varphi) \end{aligned}$$

where we used $\tilde{k}_t^L(\varphi^n) = \tilde{k}_t^{L,n}(\varphi) = 1$ and $B_L(\varphi^n) = B_L^n(\varphi)$. From the arbitrariness of $f \in \mathcal{D}_t$ we conclude that $k_t^L \equiv 1$ is the unique (in the L^2 sense) solution of (17) in $\mathbb{R}^+ \times H^\beta$. To get the negative values of t we repeat the same reasoning for the map $t \mapsto k_{-t}^L$.

Therefore, by Theorem 4.7 in [3], $\tilde{U}^L(t, \tilde{\omega})$ is unique in the sense that any other B_L -flow, $U'^L(t, \tilde{\omega})$, is such that

$$\tilde{U}^L(\cdot, \tilde{\omega}) = U'^L(\cdot, \tilde{\omega}), \quad \tilde{P} - a.e. \omega \in \tilde{\Omega}.$$

Moreover, on each compact phase space, the law of the Euler flow is a Dirac measure on the trajectories, implying that the solution is in fact deterministic. That is, we have

$$U^L(t, \varphi^L) = \varphi^L + \int_0^t B_L(U^L(s, \varphi^L)) ds, \quad \mu_\gamma^L - a.e. \varphi^L, \forall t \in \mathbb{R}$$

is the unique B_L -flow. Now, if $M \in \mathbb{N}^*$ is such that $M > L$, from $\varphi^L \equiv \varphi^M|_{[0,L]^2}$ and $B_L(t, \varphi^L) \equiv B_M(t, \varphi^M|_{[0,L]^2})$ we get

$$U^L(t, \varphi^L) \equiv U^M(t, \varphi^M|_{[0,L]^2}), \quad \forall t \in \mathbb{R}.$$

Therefore uniqueness holds for the velocity flow $\tilde{U}(t, \tilde{w})$ defined in the previous theorem which is in fact deterministic; we denote it by

$$U(t, \varphi) = \varphi + \int_0^t B(U(s, \varphi)) ds, \quad \mu_\gamma - a.e. \varphi \in H_{loc}^\beta(\mathbb{R}^2), \quad \forall t \in \mathbb{R}.$$

4.4. Invariance. The measure μ_γ is invariant under the deterministic flow $U(t, \varphi)$ defined for $t \in \mathbb{R}$ and $\varphi \in H_{loc}^\beta(\mathbb{R}^2)$. Indeed for all $f \in C_b$ we have

$$\int f d\mu_\gamma = \lim_L \int f d\mu_{L,\gamma} = \lim_L \int f(U(t, \varphi)) d\mu_{L,\gamma} = \int f(U(t, \varphi)) d\mu_\gamma, \quad \forall t \in \mathbb{R}.$$

4.5. Continuity. The flow is continuous from $H_{loc}^\beta(\mathbb{R}^2)$ to $H_{loc}^\beta(\mathbb{R}^2)$ on the support of μ_γ for all $t \in \mathbb{R}$. We write

$$\begin{aligned} \mathbb{E}_{\mu_\gamma} d_\beta(U(t, \varphi_1); U(t, \varphi_2)) &\leq \mathbb{E}_{\mu_\gamma} d_\beta(U(t, \varphi_1); U^n(t, \varphi_1)) \\ &\quad + \mathbb{E}_{\mu_\gamma} d_\beta(U^n(t, \varphi_1); U^n(t, \varphi_2)) \\ &\quad + \mathbb{E}_{\mu_\gamma} d_\beta(U^n(t, \varphi_2); U(t, \varphi_2)) \end{aligned}$$

where U^n denotes a finite dimensional approximation of U . On one hand there exist $n_1, n_2 \in \mathbb{N}$ such that for every $n \geq \max\{n_1, n_2\}$

$$\mathbb{E}_{\mu_\gamma} d_\beta(U(t, \varphi_1); U^n(t, \varphi_1)) \leq \frac{\varepsilon}{3} \quad \text{and} \quad \mathbb{E}_{\mu_\gamma} d_\beta(U^n(t, \varphi_2); U(t, \varphi_2)) \leq \frac{\varepsilon}{3}.$$

On the other, for a fixed $n \geq \max\{n_1, n_2\}$, we have that U^n is continuous; indeed it is the flow for the quadratic vector field B^n . Thus there exists a positive δ such that for $d_\beta(\varphi_1; \varphi_2) \leq \delta$ we have

$$\mathbb{E}_{\mu_\gamma} d_\beta(U^n(t, \varphi_1); U^n(t, \varphi_2)) \leq \frac{\varepsilon}{3}.$$

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REFERENCES

- [1] S. ALBEVERIO AND A. CRUZEIRO, *Global flows with invariant (Gibbs) measures for the Euler and Navier-Stokes two dimensional fluids*, Comm. Math. Phys., 129 (1990), pp. 431–444.
- [2] S. ALBEVERIO, M. R. DE FARIA, AND R. HØEGH-KROHN, *Stationary measures for the period Euler flow in two dimensions*, J. Stat. Phys., 20 (1979).
- [3] L. AMBROSIO AND A. FIGALLI, *On flows associated to Sobolev vector fields in Wiener spaces: an approach à la DiPerna-Lion*, J. Funct. Anal., 256 (2009), pp. 179–214.
- [4] N. ANTONIĆ AND K. BURAZIN, *On certain properties of spaces of locally Sobolev functions*, in Proceeding of the Conference on Applied Mathematics and Scientific Computing, 2005, pp. 109–120.
- [5] V. ARNOLD AND B. KHESIN, *Topological methods in hydrodynamics*, Springer-Verlag, 1998.
- [6] F. CIPRIANO, *The two dimensional Euler equations: a statistical study*, Comm. Math. Phys., 201 (1999), pp. 139–154.
- [7] A.-S. DE SUZZONI, *Invariant measure for the Klein-Gordon equation in a non periodic setting*. <http://arxiv.org/abs/1403.2274>, 2014.
- [8] A. MAJDA AND A. BERTOZZI, *Vorticity and incompressible flow.*, CUP, Cambridge, 2002.
- [9] C. MARCHIORO AND M. PULVIRENTI, *Mathematical theory of incompressible nonviscous fluids*, vol. 96 of Applied mathematical Sciences, Springer, 1994.

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